On the Negative Spectrum of One-Dimensional Schrödinger Operators with Point Interactions

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Abstract. We investigate negative spectra of 1–D Schrödinger operators with δ- and δ'-interactions on a discrete set in the framework of a new approach. Namely, using technique of boundary triplets and the corresponding Weyl functions, we complete and generalize the results of S. Albeverio and L. Nizhnik [3, 4]. For instance, we propose the algorithm for determining the number of negative squares of the operator with δ-interactions. We also show that the number of negative squares of the operator with δ'-interactions equals the number of negative strengths.

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1. Introduction

Consider formal differential expressions

$$\ell_{X,\alpha} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{k=1}^{\infty} \alpha_k \delta_k(x), \quad \ell_{X,\beta} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{k=-\infty}^{\infty} \beta_k \langle \cdot, \delta_k' \rangle \delta_k'(x), \tag{1.1}$$

where $\delta_k(x) := \delta(x - x_k)$ and $\delta(x)$ is a Dirack delta-function, $\alpha_k, \beta_k \in \mathbb{R}$, $k \in I$, and I equals either $\mathbb{N}$ or $\mathbb{Z}$. We assume that $X = \{x_k\}_{k\in I} \subset \mathbb{R}$ is an increasing sequence such that $d_k := x_{k+1} - x_k > 0$, $k \in I$, and

$$d_* := \inf_{k\in I} d_k > 0, \qquad d^* := \sup_{k\in I} d_k < \infty. \tag{1.2}$$

One defines the corresponding operators with δ- and δ'-interactions in $L^2(\mathbb{R})$ by the differential expression $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the domains, respectively,

$$\mathfrak{D}(L_{X,\alpha}) = \Big\{ f \in W_2^2(\mathbb{R} \setminus X) : \begin{array}{c} f(x_k+) = f(x_k-), \\ f'(x_k+) - f'(x_k-) = \alpha_k f(x_k) \end{array}, \ x_k \in X \Big\}, \quad (1.3)$$

$$\mathfrak{D}(L_{X,\beta}) = \Big\{ f \in W_2^2(\mathbb{R} \setminus X) : \begin{array}{c} f'(x_k+) = f'(x_k-), \\ f(x_k+) - f(x_k-) = \beta_k f'(x_k) \end{array}, \ x_k \in X \Big\}. \quad (1.4)$$

Note that the operators $L_{X,\alpha}$ and $L_{X,\beta}$ are self-adjoint ([1], see also [6, 9]).

Schrödinger operators with point interactions have been studied extensively in the last decades (numerous results and a comprehensive list of references may be found in [1, 2], see also Appendix K by P. Exner in [1]). In the recent publications [3, 4], S. Albeverio and L. P. Nizhnik investigated the numbers $\kappa_-(L_{X,\alpha})$ and $\kappa_-(L_{X,\beta})$ of negative eigenvalues of the operators $L_{X,\alpha}$ and $L_{X,\beta}$ in the case $|X| = n < \infty$. They described $\kappa_-(L_{X,\alpha})$ in terms of a certain continued fractions (cf. [3, Theorem 3]) and also proposed an elegant algorithm for determining $\kappa_-(L_{X,\alpha})$. In particular, they formulated necessary and sufficient conditions in terms of the distances d_k and the strengths α_k for the equalities $\kappa_-(L_{X,\alpha}) = n$ and $\kappa_-(L_{X,\alpha}) = 0$ to hold (cf. [3, Theorem 5] and [3, Theorem 4], respectively). Regarding the operators with δ'-interactions, it is shown in [4, Theorem 6] that the number of negative eigenvalues of $L_{X,\beta}$ equals n if and only if all intensities are negative, i.e., $\kappa_-(\{\beta_k\}_{k=1}^n) = n$.

In this paper, we present a new approach to investigate negative spectra of the operators with δ- and δ'-interactions on the discrete set X satisfying (1.2). Namely, we consider the operators $L_{X,\alpha}$ and $L_{X,\beta}$ as self–adjoint extensions of the symmetric operator

$$L_{\min} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad \mathfrak{D}(L_{\min}) = \mathring{W}_2^2(\mathbb{R} \setminus X), \qquad X = \{x_k\}_{k \in I} \quad (1.5)$$

and apply the technique of boundary triplets and the corresponding Weyl functions (see [12, 8] and also Section 2). We construct a boundary triplet for L_{min}^* and establish a connection between Hamiltonians $L_{X,\alpha}$ and $L_{X,\beta}$ and a certain classes of Jacobi matrices. Using this connection, we describe $\kappa_-(L_{X,\alpha})$ and $\kappa_-(L_{X,\beta})$ by means of entries of these matrices (Theorem 3.1). The latter enables us to complete and substantially generalize previous results from [3, 4] mentioned above. Namely, for a δ-type interactions, we construct an algorithm for determining $\kappa_-(L_{X,\alpha})$ (Theorem 3.5). In the case $|X| = n$, our algorithm differs from the one proposed by S. Albeverio and L.P. Nizhnik, but it is close to that (see Remark 3.11). One of our main results is the following equality $\kappa_-(L_{X,\beta}) = \kappa_-(\beta)$ (Theorem 4.1). It means that the number of negative squares of $L_{X,\beta}$ equals the number of negative intensities. In the particular case $\kappa_-(\beta) = |X| = n < \infty$, this results coincides with [4, Theorem 6]. It is interesting to mention that for the operator with δ-interactions such equality does not hold (cf. [3, 17]). We obtain sufficient condition for the inequality $\kappa_-(L_{X,\alpha}) \geq m$ (as well as for the equality) with any m (Theorem 3.3). It differs from the one recently obtained by Ogurisu in [17] and implies sufficient

condition for $\kappa_-(L_{X,\alpha}) = n$ proposed by Albeverio and Nizhnik [4, Criterion 3] in the case $\kappa_-(\alpha) = |X| = n$. In particular, the operator $L_{X,\alpha}$ with arbitrary number of negative intensities might be non-negative (see [3, Theorem 4] and also Corollary 3.6).

The results of the paper were partially announced (without proofs) in [11].

Notation. Let X be a discrete subset of $\mathbb{R}$; $|X|$ stands for the cardinal number of the set X. By $W_2^2(\mathbb{R}\setminus X)$ and $\mathring{W}_2^2(\mathbb{R}\setminus X)$ we denote the Sobolev spaces

$$W_2^2(\mathbb{R}\setminus X) := \{f \in L^2(\mathbb{R}) : \ f, f' \in AC_{loc}(\mathbb{R}\setminus X), \ f'' \in L^2(\mathbb{R})\},$$

$$\mathring{W}_2^2(\mathbb{R}\setminus X) := \{f \in W_2^2(\mathbb{R}) : f(x_k) = f'(x_k) = 0 \ \text{ for all } x_k \in X\}.$$

2. Preliminaries

Boundary triplets and closed extensions. In this subsection, we recall basic notions of the theory of boundary triplets (we refer the reader to [8, 12] for a detailed exposition).

Let A be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim\ker(A^* \pm i) \le \infty$.

Definition 2.1 ([12]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the adjoint operator A^* of A if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \ \mathfrak{D}(A^*) \to \mathcal{H}$ are linear mappings such that
(i) the second Green identity,

$$(A^*f, g)_{\mathfrak{H}} - (f, A^*g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}},$$

holds for all $f, g \in \mathfrak{D}(A^*)$, and
(ii) the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathfrak{D}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Since $n_+(A) = n_-(A)$, a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists and is not unique [12]. Moreover, $\dim\mathcal{H} = n_\pm(A)$ and $A = A^* \upharpoonright \ker(\Gamma_0) \cap \ker(\Gamma_1)$.

Any proper extension $\widetilde{A}$ of A admits the following representation (see [7])

$$\mathfrak{D}(\widetilde{A}) = \mathfrak{D}(A_{C,D}) := \mathfrak{D}(A^*) \upharpoonright \ker(D\Gamma_1 - C\Gamma_0), \quad \text{where} \quad C, \ D \in [\mathcal{H}]. \tag{2.1}$$

Note that representation (2.1) is not unique.

In what follows, we will also denote $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. Note that $A_0^* = A_0$.

Definition 2.2 ([8]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator valued function $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ defined by

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \qquad \lambda \in \rho(A_0), \quad f_\lambda \in \mathcal{N}_\lambda = \ker(A^* - \lambda), \tag{2.2}$$

is called *the Weyl function* corresponding to the boundary triplet Π.

Before formulate next result we need the following definition.

Definition 2.3 ([13]). Let $T = T^* \in \mathcal{C}(\mathfrak{H})$ and let $E_T(\lambda) = E_T(\lambda - 0)$ be the spectral function of T. Dimension of the subspace $E_T(-\infty, 0)\mathfrak{H}$ is called a number of negative squares of T and is denoted by $\kappa_-(T)$.

The Weyl function $M(\cdot)$ enables us to describe the number of negative squares of self-adjoint extensions of A.

Theorem 2.4 ([8]). *Let A be nonnegative, $A \geq 0$, and let $A_{C,D}$ be its self-adjoint extension. Assume that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* such that $A_0 = A_F$, where A_F is the Friedrichs extension of A. If the strong resolvent limit $M(0) := s - R - \lim\limits_{x \uparrow 0} M(x)$ (see* [13, Capter 8]*) exists and $M(0) \in [H]$, then*

$$\kappa_-(A_{C,D}) = \kappa_-(CD^* - DM(0)D^*). \tag{2.3}$$

The Sylvester criterion. Description of $\kappa_-(L_{X,\alpha})$ is substantially based on the following fact (see, for instance, [16, Lemma 4]).

Proposition 2.5. *Let the operator $T = T^* \in \mathcal{C}(\mathcal{H})$ admit the block-matrix representation $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $T_{11} \in [\mathcal{H}_1]$, $T_{12} = T_{12}^* \in [\mathcal{H}_2, \mathcal{H}_1]$ and $T_{22} \in \mathcal{C}(\mathcal{H}_2)$. If $0 \in \rho(T_{11})$, then*

$$\kappa_-(T) = \kappa_-(T_{11}) + \kappa_-(T_{22} - T_{21}T_{11}^{-1}T_{12}). \tag{2.4}$$

3. Operators with δ − type interactions

3.1. The case of infinite number of δ − type interactions

Consider the following Jacobi matrix in $l^2(\mathbb{N})$,

$$S = \begin{pmatrix} \alpha_1 + d_1^{-1} & -d_1^{-1} & 0 & \dots \\ -d_1^{-1} & \alpha_2 + d_1^{-1} + d_2^{-1} & -d_2^{-1} & \dots \\ 0 & -d_2^{-1} & \alpha_3 + d_2^{-1} + d_3^{-1} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3.1}$$

Notice that $S = S^*$ since $d_* = \inf_{k \in \mathbb{N}} d_k > 0$ (cf. [5, Theorem VII.1.5]).

The main result of this Section is the following description of $\kappa_-(L_{X,\alpha})$.

Theorem 3.1. *Let the set $X = \{x_k\}_{k=1}^\infty$ satisfy* (1.2)*. Let also the operator $L_{X,\alpha}$ and the matrix S be defined by* (1.3) *and* (3.1)*, respectively. Then $\kappa_-(L_{X,\alpha}) = \kappa_-(S)$.*

Proof. Consider the minimal operator (1.5). Note that $n_\pm(L_{\min}) = \infty$. Since X satisfies (1.2), the totality $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where

$$\mathcal{H} = \oplus_{k=0}^\infty \mathcal{H}_k, \qquad \Gamma_0 = \oplus_{k=0}^\infty \Gamma_0^k, \qquad \Gamma_1 = \oplus_{k=0}^\infty \Gamma_1^k, \tag{3.2}$$

$$\mathcal{H}_0 = \mathbb{C}, \qquad \Gamma_0^0 f = -f(x_1-), \qquad \Gamma_1^0 f = f'(x_1-), \quad \text{and} \tag{3.3}$$

$$\mathcal{H}_k = \mathbb{C}^2, \qquad \Gamma_0^k f = \begin{pmatrix} f(x_k+) \\ -f(x_{k+1}-) \end{pmatrix}, \qquad \Gamma_1^k f = \begin{pmatrix} f'(x_k+) \\ f'(x_{k+1}-) \end{pmatrix}, \quad k \in \mathbb{N}, \tag{3.4}$$

forms a boundary triplet for $L^*_{\min}$ [14, Lemma 1]. The corresponding Weyl function is

$$M(\lambda) = \oplus_{k=0}^{\infty} M_k(\lambda), \quad M_0(\lambda) = i\sqrt{\lambda}, \tag{3.5}$$

$$M_k(\lambda) = \begin{pmatrix} -\sqrt{\lambda}\,\mathrm{ctg}(\sqrt{\lambda}d_k) & -\sqrt{\lambda}/\sin(\sqrt{\lambda}d_k) \\ -\sqrt{\lambda}/\sin(\sqrt{\lambda}d_k) & -\sqrt{\lambda}\,\mathrm{ctg}(\sqrt{\lambda}d_k) \end{pmatrix}, \quad k \in \mathbb{N}. \tag{3.6}$$

Using (3.2)–(3.4), we obtain the representation (2.1) for $\mathfrak{D}(L_{X,\alpha})$, where

$$C = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \alpha_2 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Without loss of generality, it can be assumed that C is bounded. If there exists $\{\alpha_{k_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} |\alpha_{k_j}| = \infty$, then we put

$$\widetilde{C} = KC, \quad \widetilde{D} = KD, \quad \text{and} \quad \mathfrak{D}(L_{X,\alpha}) = \mathfrak{D}(L^*_{\min}) \upharpoonright \ker(\widetilde{D}\Gamma_1 - \widetilde{C}\Gamma_0),$$

where $K = \mathrm{diag}(1, ..., 1, \alpha_{k_1}^{-1}, 1, ..., 1, \alpha_{k_2}^{-1}, 1, ..)$.

After straightforward calculations we get the matrix $T := CD^* - DM(0)D^*$,

$$T = \begin{pmatrix} \alpha_1 + d_1^{-1} & 0 & -d_1^{-1} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ -d_1^{-1} & 0 & \alpha_2 + d_1^{-1} + d_2^{-1} & 0 & -d_2^{-1} & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -d_2^{-1} & 0 & \alpha_3 + d_2^{-1} + d_3^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

With respect to the decomposition $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$, where $\widetilde{\mathcal{H}}_1 = \mathrm{span}\{e_{2k-1}\}_{k=1}^{\infty}$ and $\widetilde{\mathcal{H}}_2 = \mathrm{span}\{e_{2k}\}_{k=1}^{\infty}$, the operator T admits the representation $T = S \oplus 0_{\mathcal{H}_2}$. Hence $\kappa_-(T) = \kappa_-(S)$, and Theorem 2.4 completes the proof. □

Using equality $\kappa_-(L_{X,\alpha}) = \kappa_-(S)$ and the following Gerschgorin theorem, we obtain sufficient condition for $\kappa_-(L_{X,\alpha}) \geq m$ as well as for the equality $\kappa_-(L_{X,\alpha}) = m$ with arbitrary finite m.

Theorem 3.2. *([15, Theorem 7.2.1]) All eigenvalues of a matrix $A = (a_{ij})_{i,j=1}^{n} \in [\mathbb{C}^n]$ are contained in the union of Gerschgorin's disks*

$$G_i = \{\lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{i \neq j} |a_{ij}|\}, \quad i \in \{1, .., n\}.$$

Theorem 3.3. *Let matrix $S = (s_{ij})_{i,j=1}^{\infty}$ be defined by (3.1). Suppose that*

$$\alpha_{k_i} < -2(d_{k_i-1}^{-1} + d_{k_i}^{-1}) \quad \textit{for} \quad k_i \in K := \{k_i\}_{i=1}^{m}. \tag{3.7}$$

Then $\kappa_-(L_{X,\alpha}) \geq m$. If, in addition, $\alpha_i > 0$ for $i \notin K$, then $\kappa_-(L_{X,\alpha}) = m$.

Proof. Consider two cases.

(a) Assume that $k_i = i,\ k_i \in K$. Denote by $S_m \in [\mathbb{C}^m]$ submatrix in the upper left corner of S. In accordance with the minimax principle (see, for instance, [10])

$$\kappa_-(S_m) \le \kappa_-(S) = \kappa_-(L_{X,\alpha}). \tag{3.8}$$

Applying Theorem 3.2 to S_m and using (3.7), we obtain $\kappa_-(S_m) = m$. Therefore $\kappa_-(L_{X,\alpha}) \ge m$ and the first assertion of the theorem holds.

Further, setting $\alpha_i = 0$ for $i \in \{1,..,m\}$, we obtain a non-negative self-adjoint operator $\widetilde{L}_{X,\alpha}$. It is obvious that $L_{X,\alpha}$ is an m-dimensional perturbation of the operator $\widetilde{L}_{X,\alpha}$. Thus from the minimax principle follows that $\kappa_-(L_{X,\alpha}) \le m$ and, consequently, the second assertion of the theorem is satisfied.

(b) Let K be an arbitrary set consisting of m natural numbers. General case is easily reduced to the previous one. Namely, there exists unitary transformation U such that

$$\widetilde{S} = U^*SU, \quad U:\, s_{k_ik_i} \to \widetilde{s}_{ii}, \quad \sum_{j\ne k_j} |s_{k_ij}| = \sum_{j\ne i} |\widetilde{s}_{ij}|,\ k_i \in K. \tag{3.9}$$

Applying previous reasoning to the matrix $\widetilde{S}$, we obtain the proof in the general case. □

Remark 3.4. Arguing as above, it is not difficult to show that $\kappa_-(L_{X,\alpha}) = \infty$ in the case of infinite m.

Theorem 3.1 enables us to obtain an algorithm for determination of $\kappa_-(L_{X,\alpha})$. Namely, define the sequence $\gamma = \{\gamma_k\}_{k=1}^\infty$ by

$$\gamma_1 := \alpha_1 + d_1^{-1}, \tag{3.10}$$

$$(i) \text{ if } \quad \gamma_k \ne 0, \quad \text{then} \quad \gamma_{k+1} := \alpha_{k+1} + d_{k+1}^{-1} + d_k^{-1} - d_k^{-2}\gamma_k^{-1}, \quad k \ge 1; \tag{3.11}$$

$$(ii) \text{ if } \quad \gamma_k = 0, \quad \text{then} \quad \begin{array}{c} \gamma_{k+1} := \infty \\ \gamma_{k+2} := \alpha_{k+2} + d_{k+1}^{-1} + d_{k+2}^{-1} \end{array}, \quad k \ge 1. \tag{3.12}$$

Theorem 3.5. *Let the set $X = \{x_k\}_{k=1}^\infty$ satisfy* (1.2). *Let the operator $L_{X,\alpha}$ be defined by* (1.3) *and let the sequence $\gamma = \{\gamma_k\}_{k=1}^\infty$ be defined by* (3.10)–(3.12). *Then*

$$\kappa_-(L_{X,\alpha}) = \kappa_-(\gamma) + N_\infty(\gamma),$$

where $\kappa_-(\gamma)$ and $N_\infty(\gamma)$ are the number of negative and infinite elements, respectively, in the sequence γ.

Proof. Consider two cases.

(a) Let $\gamma_1 = \alpha_1 + d_1^{-1} \ne 0$. Setting $T_{11} := \gamma_1 I_{\mathbb{C}}$ and applying Proposition 2.5 to the matrix (3.1), we get $\kappa_-(S) = \kappa_-(\gamma_1) + \kappa_-(S_2)$, where

$$S_2 := \begin{pmatrix} \gamma_2 & -d_2^{-1} & 0 & \dots \\ -d_2^{-1} & \alpha_3 + d_2^{-1} + d_3^{-1} & -d_3^{-1} & \dots \\ 0 & -d_3^{-1} & \alpha_4 + d_3^{-1} + d_4^{-1} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Further, if $\gamma_2 \neq 0$, then we set $T_{11} = \gamma_2 I_{\mathbb{C}}$ and apply Proposition 2.5 to the matrix S_2. Thus if $\gamma_k \neq 0$ for all $k \in \mathbb{N}$, i.e., $N_\infty(\gamma) = 0$, then we obtain $\kappa_-(S) = \kappa_-(\gamma)$.

(b) Assume that $\gamma_1 = \alpha_1 + d_1^{-1} = 0$. Then $\gamma_2 = \infty$ and $\gamma_3 = \alpha_3 + d_2^{-1} + d_3^{-1}$. Let $T_{11} := \begin{pmatrix} 0 & -d_1^{-1} \\ -d_1^{-1} & \alpha_2 + d_1^{-1} + d_2^{-1} \end{pmatrix} \in [\mathbb{C}^2]$. Since $\det T_{11} = -d_1^{-2} \neq 0$, by Proposition 2.5, we get $\kappa_-(S) = \kappa_-(T_{11}) + \kappa_-(S_3)$, where

$$S_3 := \begin{pmatrix} \gamma_3 & -d_3^{-1} & 0 & \dots \\ -d_3^{-1} & \alpha_4 + d_3^{-1} + d_4^{-1} & -d_4^{-1} & \dots \\ 0 & -d_4^{-1} & \alpha_5 + d_4^{-1} + d_5^{-1} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Since $\kappa_-(T_{11}) = 1$, we get $\kappa_-(S) = N_\infty(\{\gamma_1, \gamma_2\}) + \kappa_-(S_3)$.

Proceeding as above, we obtain the desired result. $\square$

Following [4], consider continued fraction $A_k := [\alpha_k; d_{k-1}, \alpha_{k-1}, .., \alpha_1]$. It is easy to verify by induction that if $\gamma_k \neq 0$ for all $\gamma_k \in \gamma$, then

$$\gamma_k = d_k^{-1} + A_k, \quad k \geq 1. \tag{3.13}$$

Theorem 3.5 and equality (3.13) yield the following result.

Corollary 3.6. *The operator $L_{X,\alpha}$ is non-negative if and only if*

$$A_k > -d_k^{-1}, \quad k \geq 1.$$

3.2. The case of finite number of δ − type interactions

Setting $\alpha_k = 0,\ k > n$, in (1.3), we obtain the operator with δ-interactions on a finite set. Using Theorems 3.1 and 3.5, we obtain the following description of the negative squares $\kappa_-(L_{X,\alpha})$. Namely, define the sequence

$$\widetilde{\gamma}_1 := \alpha_1 + d_1^{-1}, \tag{3.14}$$

$$(i) \text{ if } \widetilde{\gamma}_k \neq 0, \text{ then } \widetilde{\gamma}_{k+1} := \begin{cases} \alpha_{k+1} + d_{k+1}^{-1} + d_k^{-1} - d_k^{-2}\widetilde{\gamma}_k^{-1}, & k \leq n-1, \\ d_{k+1}^{-1} + d_k^{-1} - d_k^{-2}\widetilde{\gamma}_k^{-1}, & k \geq n; \end{cases} \tag{3.15}$$

$$(ii) \text{ if } \widetilde{\gamma}_k = 0, \text{ then } \widetilde{\gamma}_{k+1} := \infty, \quad \widetilde{\gamma}_{k+2} := \begin{cases} \alpha_{k+2} + d_{k+1}^{-1} + d_{k+2}^{-1}, & k \leq n-2, \\ d_{k+1}^{-1} + d_{k+2}^{-1}, & k \geq n-1. \end{cases} \tag{3.16}$$

Corollary 3.7. *Let $X = \{x_k\}_{k=1}^n \subset \mathbb{R}$ be a finite set. Let also the operator $L_{X,\alpha}$ be defined by* (1.3) *and let $\widetilde{\gamma} = \{\widetilde{\gamma}_k\}_{k=1}^\infty$ be the sequence defined by* (3.14)–(3.16). *Then*

$$\kappa_-(L_{X,\alpha}) = \kappa_-(\widetilde{\gamma}) + N_\infty(\widetilde{\gamma}).$$

Corollary 3.7 has one essential drawback. To obtain $\kappa_-(L_{X,\alpha})$, we must find infinite number of elements $\widetilde{\gamma}_n,\ n \in \mathbb{N}$. But it is possible to overcome this by

treating $L_{X,\alpha}$ as an extension of the minimal operator with finite deficiency indices. Namely, define the matrix $S \in \mathbb{C}^{n\times n}$,

$$S = \begin{pmatrix} \alpha_1 + d_1^{-1} & -d_1^{-1} & 0 & \dots & 0 \\ -d_1^{-1} & \alpha_2 + d_1^{-1} + d_2^{-1} & -d_2^{-1} & \dots & 0 \\ 0 & -d_2^{-1} & \alpha_3 + d_2^{-1} + d_3^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_n + d_{n-1}^{-1} \end{pmatrix}. \tag{3.17}$$

Theorem 3.8. *Let $X = \{x_k\}_{k=1}^n \subset \mathbb{R}$ be a finite set. Let the operator $L_{X,\alpha}$ be defined by* (1.3) *and let S be the matrix* (3.17)*. Then $\kappa_-(L_{X,\alpha}) = \kappa_-(S)$.*

Proof. Consider the operator $L_{\min}$ of the form (1.5) with $X = \{x_k\}_{k=1}^n$. Note that $n_\pm(L_{\min}) = 2n$. The boundary triplet for $L_{\min}^*$ might be defined by (cf. [12, Section III,§1])

$$\mathcal{H} = \oplus_{k=0}^n \mathcal{H}_k, \quad \Gamma_0 = \oplus_{k=0}^n \Gamma_0^k, \quad \Gamma_1 = \oplus_{k=0}^n \Gamma_1^k, \quad \text{where} \tag{3.18}$$

$$\mathcal{H}_0 = \mathbb{C}, \quad \Gamma_0^0 f = -f(x_1-), \quad \Gamma_1^0 f = f'(x_1-), \tag{3.19}$$

$$\mathcal{H}_k = \mathbb{C}^2,\ \Gamma_0^k f = \begin{pmatrix} f(x_k+) \\ -f(x_{k+1}-) \end{pmatrix},\ \Gamma_1^k f = \begin{pmatrix} f'(x_k+) \\ f'(x_{k+1}-) \end{pmatrix}, k \in \{1,..,n-1\}, \tag{3.20}$$

$$\mathcal{H}_n = \mathbb{C}, \quad \Gamma_0^n f = f(x_n+), \quad \Gamma_1^n f = f'(x_n+). \tag{3.21}$$

The corresponding Weyl function $M(\lambda)$ is

$$M(\lambda) = \oplus_{k=0}^n M_k(\lambda), \quad M_0(\lambda) = M_n(\lambda) = i\sqrt{\lambda} \quad \text{and}$$

$M_k(\lambda)$ for $k \in \{1,..,n-1\}$ is given by (3.6).

Using (3.18)–(3.21), we obtain a description of $\mathfrak{D}(L_{X,\alpha})$ in the form (2.1), where

$$C = \begin{pmatrix} 0 & \alpha_1 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \alpha_n \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix}, D = \begin{pmatrix} -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Further, it easy to verify that the matrix $T = CD^* - DM(0)D^*$ has the form

$$T = \begin{pmatrix} \alpha_1 + d_1^{-1} & 0 & -d_1^{-1} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ -d_1^{-1} & 0 & \alpha_2 + d_1^{-1} + d_2^{-1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \alpha_n + d_n^{-1} \end{pmatrix}.$$

Arguing as in the proof of Theorem 3.1, we complete the proof. □

Define the sequence $\gamma = \{\gamma_k\}_{k=1}^n$ as follows

$$\gamma_1 := \alpha_1 + d_1^{-1}, \tag{3.22}$$

$$(i) \text{ if } \gamma_k \neq 0, \text{ then } \quad \gamma_{k+1} := \begin{cases} \alpha_{k+1} + d_{k+1}^{-1} + d_k^{-1} - d_k^{-2}\gamma_k^{-1}, & k \leq n-2 \\ \alpha_k + d_{k-1}^{-1} - d_{k-1}^{-2}\gamma_{k-1}^{-1}, & k = n-1 \end{cases}; \tag{3.23}$$

$$(ii) \text{ if } \gamma_k = 0, \text{ then } \quad \begin{array}{c} \gamma_{k+1} := \infty, \quad k \in \{1,..,n-1\}, \\ \gamma_{k+2} := \alpha_{k+2} + d_{k+1}^{-1} + d_{k+2}^{-1}, \quad k \in \{1,..,n-2\}. \end{array} \tag{3.24}$$

Theorem 3.9. *Assume $X = \{x_k\}_{k=1}^n$. Let the operator $L_{X,\alpha}$ be defined by (1.3) and let the sequence $\gamma = \{\gamma_k\}_{k=1}^n$ be defined by (3.22)–(3.24). Then*

$$\kappa_-(L_{X,\alpha}) = \kappa_-(\gamma) + N_\infty(\gamma).$$

We omit the proof since it is analogous to that of Theorem 3.5.

Proposition 3.10. *Corollary 3.7 and Theorem 3.9 are equivalent, i.e.,*

$$\kappa_-(\widetilde{\gamma}) + N_\infty(\widetilde{\gamma}) = \kappa_-(\gamma) + N_\infty(\gamma),$$

where $\widetilde{\gamma} = \{\widetilde{\gamma}_n\}_{n=1}^\infty$ and $\gamma = \{\gamma_n\}_{n=1}^\infty$ are defined by (3.14)–(3.16) *and* (3.22)–(3.24)*, respectively.*

Proof. Since $\widetilde{\gamma_k} = \gamma_k$ for $k < n$, it suffices to verify that

$$\kappa_-(\gamma_n) + N_\infty(\gamma_n) = \kappa_-(\{\widetilde{\gamma_k}\}_{k=n}^\infty) + N_\infty(\{\widetilde{\gamma_k}\}_{k=n}^\infty). \tag{3.25}$$

First, assume that $\widetilde{\gamma}_m < 0$ for some $m \geq n$. Then, by (3.15), $\widetilde{\gamma}_{m+1} > d_{m+1}^{-1}$ and hence

$$\widetilde{\gamma}_k \geq d_k^{-1}, \text{ for all } k > m. \tag{3.26}$$

The latter also yields that in this case $\kappa_-(\{\widetilde{\gamma_k}\}_{k=n}^\infty) \leq 1$ and $N_\infty(\{\widetilde{\gamma_k}\}_{k=m}^\infty) = 0$.

Further, if $\widetilde{\gamma}_m = 0$ for some $m > n$, then, by (3.16), $\widetilde{\gamma}_{m+1} = \infty$ and $\widetilde{\gamma}_{m+i} = d_{m+i}^{-1} + d_{m+i-1}^{-1}$, $i \geq 2$. Therefore, $N_\infty(\{\widetilde{\gamma_k}\}_{k=n}^\infty) \leq 1$ and $\kappa_-(\{\widetilde{\gamma_k}\}_{k=m}^\infty) = 0$.

Consider three cases.

(a) Let $\gamma_n \geq 0$. Combining (3.23) with (3.15), we get $\widetilde{\gamma}_n = \gamma_n + d_n^{-1} \geq d_n^{-1}$. By (3.15), $\widetilde{\gamma}_k$ satisfies (3.26) with $m = n$, and hence (3.25) clearly holds.

(b) Let $\gamma_n = \infty$. Then $\gamma_{n-1} = \widetilde{\gamma}_{n-1} = 0$ and $\widetilde{\gamma}_n = \infty$. Thus $\widetilde{\gamma}_k$ satisfies (3.26) for all $k \geq n$, and hence (3.25) holds.

(c) Assume now that $\gamma_n < 0$.

If $\widetilde{\gamma}_k = 0$ for some $k \geq n$, then arguing as above we arrive at (3.25).

Suppose that $\widetilde{\gamma}_k \neq 0$, $k \geq n$. To prove (3.25) it suffices to show that $\widetilde{\gamma}_{n+i} < 0$ for some $i > 0$. Assume the converse, i.e., $\widetilde{\gamma}_k > 0$ for all $k \geq n$. Denote $\xi_k := d_k^{-1} - \widetilde{\gamma}_k$, $k \geq n$. Clearly, $\xi_k < d_k^{-1}$ for all $k \geq n$. Note that, by (3.15), $\xi_{k+1} = d_k^{-1}\xi_k(d_k^{-1} - \xi_k)^{-1}$. Further, the inequality $0 < \xi_n < d_n^{-1}$ holds since $\gamma_n < 0$. Moreover, $0 < d_n^{-1} - \xi_n < d_n^{-1} < d_*^{-1}$ (see (1.2)) and hence

$$\xi_{n+1} = d_n^{-1}\xi_n(d_n^{-1} - \xi_n)^{-1} = \xi_n + \xi_n^2(d_n^{-1} - \xi_n)^{-1} > \xi_n + \xi_n^2 d_*.$$

Similarly, $\xi_{n+1} < d_{n+1}^{-1} \le d_*^{-1}$ yields

$$\xi_{n+2} > \xi_{n+1} + \xi_{n+1}^2 d_* > \xi_n + \xi_n^2 d_* + \xi_n^2 d_* = \xi_n + 2\,\xi_n^2 d_*.$$

Therefore, we get $\xi_{n+i} > \xi_n + i\,\xi_n^2 d_*$, $i \in \mathbb{N}$. Hence there exists $i_0 \in \mathbb{N}$ such that

$$\xi_n + i\xi_n^2 d_* > d_*^{-1} > d_{n+i}^{-1}, \qquad i \ge i_0.$$

Therefore we get $\xi_{n+i_0} > d_{n+i_0}^{-1}$ and consequently $\widetilde{\gamma}_{n+i_0} < 0$. This contradiction comletes the proof of (3.25).

Combining **(a)**, **(b)**, and **(c)**, we arrive at the desired result. □

Remark 3.11. In [3], S. Albeverio and L. P. Nizhnik obtained another description of $\kappa_-(L_{X,\alpha})$. Namely, define the function φ as a solution of the problem

$$\varphi''(x) = 0, \quad x \notin X, \qquad \varphi(x) \equiv 1, \quad x < x_1, \quad \text{and} \tag{3.27}$$

$$\varphi(x_k+) = \varphi(x_k-), \quad \varphi'(x_k+) - \varphi'(x_k-) = \alpha_k \varphi(x_k) \quad \text{for} \quad x_k \in X. \tag{3.28}$$

Theorem 3 from [3] states that $\kappa_-(L_{X,\alpha})$ *equals the signature of the sequence*

$$(\varphi(x_1),\, \varphi(x_2), ..., \varphi(x_n),\, (1 + \alpha_n d_{n-1})\varphi(x_n) - \varphi(x_{n-1})). \tag{3.29}$$

Note that this result may be deduced from Theorem 3.8 and vise versa. Namely, let Δ_k be a k-th order leading principle minor of the matrix S defined by (3.17). Then one can check that

$$\Delta_k = \frac{\varphi(x_{k+1})}{d_{k-1} \cdot ... \cdot d_1}, \qquad k \in \{1, .., n\}.$$

4. Operators with δ'-interactions

The main result of this Section is the following theorem.

Theorem 4.1. *Let* $X = \{x_k\}_{k=-\infty}^{\infty}$ *be a discrete subset of* $\mathbb{R}$ *satisfying* (1.2)*,* $L_{X,\beta}$ *the operator defined by* (1.4)*, and* $\beta = \{\beta_k\}_{k=-\infty}^{\infty} \subset \mathbb{R}$*. Then* $\kappa_-(L_{X,\beta}) = \kappa_-(\beta)$*.*

Proof. We divide the proof into several steps.

(a) Consider the minimal operator $L_{\min}$ (1.5). Since X satisfies (1.2), we can choose the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $L_{\min}^*$ as follows [14, Lemma 1]

$$\mathcal{H} = \oplus_{k=-\infty}^{\infty} \mathcal{H}_k, \quad \Gamma_0 f = \oplus_{k=-\infty}^{\infty} \Gamma_0^k f, \quad \Gamma_1 f = \oplus_{k=-\infty}^{\infty} \Gamma_1^k f, \tag{4.1}$$

where $\Pi_k = \{\mathcal{H}_k, \Gamma_0^k, \Gamma_1^k\}$, $k \in \mathbb{Z}$, is given by (3.4).

The corresponding Weyl function is $M(\lambda) = \oplus_{k=-\infty}^{\infty} M_k(\lambda)$ with $M_k(\lambda)$ defined by (3.6).

The domain of the operator $L_{X,\beta}$ admits the representation $\mathfrak{D}(L_{X,\beta}) = \mathfrak{D}(L^*_{\min}) \upharpoonright \ker(D\Gamma_1 - C\Gamma_0)$ with D and C determined, respectively, by

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -1 & 1 & 0 & 0 & \dots \\ \dots & 0 & \ulcorner \beta_0 & 0 \urcorner & 0 & \dots \\ \dots & 0 & \llcorner\, 0 & -1 \lrcorner & 1 & \dots \\ \dots & 0 & 0 & 0 & \beta_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \text{ and } \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & \ulcorner 1 & 0 \urcorner & 0 & \dots \\ \dots & 0 & \llcorner 0 & 0 \lrcorner & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Arguing as in the proof of Theorem 3.1, we assume that D is bounded. After straightforward calculations we get the operator $T = CD^* - DM(0)D^*$,

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \beta_0 d_{-1}^{-1} & -d_{-1}^{-1} & 0 & 0 & 0 & \dots \\ \dots & \ulcorner \beta_0 + {\beta_0}^2 d_{-1}^{-1} & -\beta_0 d_{-1}^{-1} \urcorner & 0 & 0 & 0 & \dots \\ \dots & \llcorner -\beta_0 d_{-1}^{-1} & d_{-1}^{-1} + d_0^{-1} \lrcorner & \beta_1 d_0^{-1} & -d_0^{-1} & 0 & \dots \\ \dots & 0 & \beta_1 d_0^{-1} & \ulcorner \beta_1 + {\beta_1}^2 d_0^{-1} & -\beta_1 d_0^{-1} \urcorner & 0 & \dots \\ \dots & 0 & -d_0^{-1} & \llcorner -\beta_1 d_0^{-1} & d_0^{-1} + d_1^{-1} \lrcorner & \beta_2 d_1^{-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{4.2}$$

By Theorem 2.4, $\kappa_-(L_{X,\beta}) = \kappa_-(T)$.

(b) Note that the matrix (4.2) admits the representation $T = A + B$, where

$$A = \sum_{k=-\infty}^{\infty} \beta_k (\cdot, e_{2k-1}) e_{2k-1}, \quad B = \sum_{k=-\infty}^{\infty} d_{k-1}^{-1} (\cdot, \mathbf{b}_k) \mathbf{b}_k,$$

with $\mathbf{b}_k := e_{2k-1} + \beta_k e_{2k} - e_{2k+1}$. Since $d_k^{-1} > 0$, one gets

$$\kappa_-(T) \ \leq \ \kappa_-(A). \tag{4.3}$$

(c) Let $s \in \mathbb{Z}_- \cup \{0\}$ and $r \in \mathbb{N}$. Consider the matrix $T_{s,r} \in [\mathbb{C}^{2(r-s)+1}]$,

$$T_{s,r} := A_{s,r} + B_{s,r}, \quad \text{where} \quad A_{s,r} = \sum_{k=s-1}^{r+1} \beta_k (\cdot, e_{2k}) e_{2k} \text{ and}$$

$$B_{s,r} = \sum_{k=s+1}^{r-1} d_{k-1}^{-1} (\cdot, \mathbf{b}_k) \mathbf{b}_k + d_{s-1}^{-1} (\cdot, \mathbf{y}_s) \mathbf{y}_s + d_{r-1}^{-1} (\cdot, \mathbf{x}_r) \mathbf{x}_r,$$

with $\mathbf{y}_s := \beta_s e_{2s-2} + e_{2s-1}$ and $\mathbf{x}_r := e_{2r+1} + \beta_r e_{2r+2}$.

It is clear that $\operatorname{ran}(A_{s,r}) \cap \operatorname{ran}\left(B_{s,r} - d_{s-1}^{-1}(\cdot, \mathbf{y}_s)\mathbf{y}_s\right) = \{0\}$ and hence

$$\kappa_-(T_{s,r} - d_{s-1}^{-1}(\cdot, \mathbf{y}_s)\mathbf{y}_s) = \kappa_-(A_{s,r}).$$

According to the choice of the matrix $T_{s,r}$, we get

$$\kappa_-(T) \geq \kappa_-(T_{s,r} - d_{s-1}^{-1}(\cdot, \mathbf{y}_s)\mathbf{y}_s) - \operatorname{rank}(d_{s-1}^{-1}(\cdot, \mathbf{y}_s)\mathbf{y}_s) = \kappa_-(A_{s,r}) - 1. \tag{4.4}$$

Combining (4.3) with (4.4), we obtain $\kappa_-(A) - 1 \leq \kappa_-(T) \leq \kappa_-(A)$.

If $\kappa_-(\beta)=\infty$, then (4.3) yields $\kappa_-(T)=\infty$. Therefore, $\kappa_-(L_{X,\beta})=\infty$ and theorem is proven in the case $\kappa_-(\beta)=\infty$.

(d) Assume now that $\kappa_-(\beta)=m<\infty$. Let us show that

$$\det(T_{s,r})=\big(x_r-x_{s-1}+\sum_{k=s}^{r}\beta_k\big)\prod_{k=s}^{r}d_{k-1}^{-1}\beta_k. \tag{4.5}$$

For $s=0$, $r=1$ equality (4.5) is obvious. Suppose that (4.5) holds with $s+1<0$ and $r>1$. Note that the second row $\mathbf{t}_2$ of the matrix $T_{s,r}$ admits a decomposition

$$\mathbf{t}_2=\mathbf{t}_2^1+\mathbf{t}_2^2,\ \text{where}$$
$$\mathbf{t}_2^1=\begin{pmatrix}-\beta_s d_{s-1}^{-1} & d_{s-1}^{-1} & 0 & \dots & 0\end{pmatrix}\ \text{and}\ \mathbf{t}_2^2=\begin{pmatrix}0 & d_s^{-1} & \beta_{s+1}d_s^{-1} & \dots & 0\end{pmatrix}.$$

Then $\det(T_{s,r})=\det(T_{s,r}^1)+\det(T_{s,r}^2)$, where $T_{s,r}^1$ and $T_{s,r}^2$ are matrices obtained by replacement of $\mathbf{t}_2$ in $T_{s,r}$ by $\mathbf{t}_2^1$ and $\mathbf{t}_2^2$, respectively. Adding to the first row of $T_{s,r}^1$ the second row multiplied by β_s, we arrive at the equality

$$\det(T_{s,r}^1)=\beta_s d_{s-1}^{-1}\det(T_{s+1,r}). \tag{4.6}$$

It is easily seen that $\det(T_{s,r}^2)=(\beta_s+\beta_s^2 d_{s-1}^{-1})\det(T^2_{2(r-s)})$, where $T^2_{2(r-s)}\in[\mathbb{C}^{2(r-s)}]$ is an algebraic complement of $\beta_s+\beta_s^2 d_{s-1}^{-1}$. Adding to the second row of $T^2_{2(r-s)}$ the first row multiplied by $-\beta_{s+1}$ and to the third row the first row, we get $\det(T_{s,r}^2)=(\beta_s+\beta_s^2 d_{s-1}^{-1})\beta_{s+1}d_s^{-1}\det(T^2_{2(r-s-1)})$. Proceeding analogously, one, finally, obtains

$$\det(T_{s,r}^2)=(\beta_s+\beta_s^2 d_{s-1}^{-1})\big(\prod_{k=s+1}^{r}d_{k-1}^{-1}\beta_k\big). \tag{4.7}$$

Combining (4.6) and (4.7), we arrive at (4.5).

Since $\kappa_-(\beta)<\infty$ and the difference $(x_r-x_{s-1})=\sum_{k=s}^{r}d_k$ is unbounded as either $-s$ or r tends to infinity, for sufficiently large s and r we have $\big(x_r-x_{s-1}+\sum_{k=s}^{r}\beta_k\big)>0$ and $\beta_k>0$ for $k<s$ or $k>r$. Hence

$$\operatorname{sgn}(\det(T_{s,r}))=\operatorname{sgn}\big(\prod_{k=s}^{r}d_{k-1}^{-1}\beta_k\big)=(-1)^{\kappa_-(\beta)}=(-1)^m\neq(-1)^{m-1}.$$

Therefore $\kappa_-(T_{s,r})=m$ and, by (4.3), we, finally, get

$$m=\kappa_-(T_{s,r})\le\kappa_-(T)\le\kappa_-(A)=m,$$

i.e., $\kappa_-(T)=m$. The proof is completed. □

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References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics*, Sec. Edition, AMS Chelsea Publ., 2005.

[2] S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators and Schrödinger Type Operators*, Cambridge Univ. Press, 2000.

[3] S. Albeverio, L.Nizhnik, *On the Number of Negative Eigenvalues of One-Dimensional Schrodinger Operator with Point Interactions*, Lett. Math. Phys. **65** (2003), 27–35.

[4] S. Albeverio, L. Nizhnik, *Schrodinger Operators with a number of Negative Eigenvalues equal to the number of Point Interactions*, Methods Funct. Anal. Topology, **9** (2003), no.4 , 273–286.

[5] Yu.M. Berezanskii, *Expansions in eigenfunctions of selfadjoint operators*, Naukova Dumka, Kiev, 1965.

[6] D. Buschmann, G. Stolz, J. Weidmann, *One-dimensional Schrödinger operators with local point interactions*, J. Reine Angew. Math., 1995, 169–186.

[7] V.A. Derkach, S. Hassi, M.M. Malamud and H.S.V. de Snoo *Generalized resolvents of symmetric operators and admissibility*, Methods Funct. Anal. Topology, **6** (2000), no.3, 24-55.

[8] V.A. Derkach, M.M. Malamud, *Generalised Resolvents and the boundary value problems for Hermitian Operators with gaps*, J. Funct. Anal., **95** (1991), no.1, 1–95.

[9] F. Gesztezy, W. Kirsch, *One-dimensional Schrödinger operators with interactions singular on a discrete set*, J. Reine Angew. Math., **362** (1985), 27–50.

[10] I.M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* [in Russian], Fizmatgiz, Moscow, 1963.

[11] N.I. Goloshchapova, L.L. Oridoroga, *The One-Dimensional Schödinger Operator with Point δ- and δ'-interactions*, Math. Notes, **84** (2008), no.1, 125–129.

[12] V.I. Gorbachuk, M.L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Mathematics and its Applications (Soviet Series) 48, Kluwer Academic Publishers Group, Dordrecht, 1991.

[13] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.

[14] A.N. Kochubei, *One-Dimensional Point Interactions*, Ukrainian Math. J., **41** (1989), no.10, 90–95.

[15] P. Lancaster, *Theory of Matrices* [Russian translation], Nauka, Moscow, 1973 (English: Academic Press, New York, 1969).

[16] M.M. Malamud, *On certain classes of Extensions of Hermitian Operators with gaps*, Ukrainian Math. J., **44** (1992), no.2, 215–233.

[17] O. Ogurisu, *On the Number of Negative Eigenvalues of a Schrodinger Operator with Point Interactions*, Letters in Mathematical Physics, **85** (2008), 129–133.

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